

EXPANDING RICCI SOLITONS WITH PINCHED RICCI CURVATURE

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ABSTRACT. In this paper, we prove that expanding gradient Ricci solitons with (positively) pinched Ricci curvature are trivial ones. Namely, they are either compact or flat.

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1. INTRODUCTION

In this paper we consider Problem 9.62 in the famous book [2], which is also the unanswered question in [6]. Namely, when $n \geq 3$, do there exist expanding gradient Ricci solitons with (positively) pinched Ricci curvature? In dimension three, we settle the problem completely. Here we recall that the (positively) pinched Ricci curvature for the Riemannian manifold (M^n, g) is in the sense that

$$(1) \quad Rc \geq \epsilon Rg \geq 0,$$

where R and Rc are the scalar and Ricci curvatures of the metric g respectively, $\epsilon > 0$ is a small constant. This concept plays an important role in the seminal work of R. Hamilton [3]. We remark that the compact expanding gradient Ricci solitons are Einstein. This result is known in G. Perelman [7]. Then we may assume that the expanding gradient Ricci soliton $(M^n, g(t), \phi)$ under consideration is complete, non-compact, and in canonical form that

$$D^2\phi = Rc + \frac{1}{2t}g, \quad t > 0.$$

We denote by $d_g(x, o)$ the distance between the points x and o in (M, g) .

We show that there are only trivial ones.

Theorem 1. *Expanding gradient Ricci solitons (M, g) with (positively) pinched Ricci curvature and curvature decay at the order $d_g(x, o)^{-2-\delta}$ for some $\delta > 0$, are trivial ones. Namely, they are either compact or R^n with flat metric.*

We remark that in dimension three, the curvature decay condition is automatically true [6]. This result is used in [5]. In the dimension bigger than three, the same is true for locally conformally flat expanding gradient

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Ricci solitons with pinched Ricci curvature. In general, since we are studying the Ricci flow, we should have the curvature decay order as that of the Ricci curvature. This will be considered in the future.

This paper is organized as follows. In section 2 we recall some famous results, which will be in use in section 3. Theorem 1 is proved in section 3.

We shall use r denote various uniform positive constants.

2. PRELIMINARY

Before we prove our main result Theorem 1, we cite the following results, which will be in use in next section. The first is

Proposition 2. (*Hamilton, Proposition 9.46 in [2]*). *If $(M, g(t))$, $t > 0$, is a complete non-compact expanding gradient Ricci soliton with $Rc > 0$, then*

$$AVR(g(t)) := \lim_{r \rightarrow \infty} \frac{B_{g(t)}(o, r)}{r^n} > 0,$$

where the definition of $AVR(g(t))$ is independent of the base point $o \in M$.

The second is

Proposition 3. ([6]). *If $(M, g(t))$, $t > 0$, is a complete non-compact expanding gradient Ricci soliton with (1), then the scalar curvature is quadratic exponential decay.*

The third one is Theorem 1.1 in [1]. Roughly speaking, the result says that for the complete and non-compact (M, g) , if $AVR(g) > 0$ and the curvature decay suitably, then it is an asymptotic manifold. We invite the readers to papers [1] and [4] for the definitions of asymptotic flat manifold M_τ and coordinates $(z) = (z^i)$ at infinity (also called the asymptotic coordinates).

The last one is Proposition 10.2 in [4]. Namely,

Proposition 4. ([4]). *If (M, g) is asymptotic flat with $g \in \mathbf{M}_\tau$, for some $\tau > \frac{n-2}{2}$, and the Ricci curvature is non-negative. Then the mass*

$$m(g) := \lim_{r \rightarrow \infty} \int_{S_r} \mu \lrcorner dz$$

is non-negative, with $m(g) = 0$ if and only if (M, g) is isometric to R^n with its Euclidean metric. Here $S_r : \{x \in M; d_g(o, x) = r\}$,

$$\mu = (\partial_i g_{ij} - \partial_j g_{ii}) \partial_j, \quad \partial_j = \frac{\partial}{\partial z^j},$$

and (z^j) are the asymptotic coordinates.

3. PROOF OF THEOREM 1

We argue by contradiction. So we assume that $(M, g(t))$ is not flat.

Using the strong maximum principle [8] to the Ricci soliton, we may assume that the scalar curvature is positive, i.e., $R > 0$. Hence we know that $Rc > 0$. According to the arguments in [2] and [6], we know that ϕ is a

proper strict convex function, which implies by using the Morse theory that M^n is diffeomorphic to R . Using Proposition 2, Proposition 3, and Theorem 1.1 in [1] we know that $(M, g(t))$ is an asymptotic flat manifold. We also know that

$$\phi(x) \approx d_g(x, o)^2, \quad |\nabla \phi(x)| \approx d_g(x, o).$$

Recall that in coordinates (z^j) at infinity, we have the Ricci soliton equation

$$\phi_{ij} = R_{ij} + \frac{1}{2t}g_{ij}.$$

For notation simple, we let $t = 1/2$. Then we have $g_{ij} = \phi_{ij} - R_{ij}$. By Ricci pinching condition we know that R_{ij} decay exponentially and

$$\Delta \phi = \phi_{ii} = R + n.$$

Recall the Ricci formula

$$\phi_{iji} = \phi_{iij} + R_{ji}\phi_i.$$

This also implies that

$$\nabla R = -2Rc(\nabla \phi).$$

Hence $|\nabla R|$ decays in the exponent rate.

We now compute the mass. Using the Ricci formula, we have

$$\begin{aligned} m(g) &= \lim_{r \rightarrow \infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) \partial_j \lrcorner dz \\ &= \lim_{r \rightarrow \infty} \int_{S_r} (\partial_i \phi_{ij} - \partial_j \phi_{ii}) \partial_j \lrcorner dz \\ &= \lim_{r \rightarrow \infty} \int_{S_r} \phi_{iji} \partial_j \lrcorner dz \\ &= \lim_{r \rightarrow \infty} \int_{S_r} (\phi_{iij} + R_{ji}\phi_i) \partial_j \lrcorner dz \\ &= \lim_{r \rightarrow \infty} \int_{S_r} R_j \partial_j \lrcorner dz \\ &= 0. \end{aligned}$$

Using Proposition 4 we know that $(M, g(1/2))$ is R^n with the Euclidean metric. A contradiction. This completes the proof of Theorem 1. q.e.d.

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